ANALYSIS OF STEADY TWO-DIMENSIONAL FLOW OF NEWTONIAN FLUID IN A CONSTRICTED ARTERY WITH HEAT TRANSFER BY COMPARISON

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ABSTRACT: *The analysis of steady state blood flow with heat transfer in an axisymmetric artery having constriction of cosine shape is presented. The blood is assumed to behave as an incompressible viscous fluid. The governing Navier-Stokes equations are transformed into compatibility equation along with energy equation. These non-linear equations are solved analytically with the help of Adomian decomposition method (ADM) and Regular perturbation method (RPM). The analytical results thus obtained are presented graphically for stream lines, wall shear stress, separation, reattachment points and temperature distribution. It is observed that an increase in constriction height (* ε *) increases the wall shear stress, temperature and decreases the critical Reynolds number (* Re *). A parametric study of blood flow behavior is presented by two methods and comparison of two methods shows that ADM is reliable, easily computable and provides faster convergent series.*

Keywords: Newtonian fluid, ADM, RPM, wall shear stress, heat transfer.

1. INTRODUCTION

Congenital heart disease usually manifests itself in childhood but may pass unnoticed manhood. Arterial defects which are well tolerated may cause no symptoms until adult life or may first be detected incidentally on routine examination. Such patients may remain in good health for many years and subsequently develop related problems. In normal human life heart pumps blood producing oscillatory flow in vessel. It is well known that the deposit of cholesterol and proliferation of connective tissue may be responsible for the abnormal growths in the lumen of the artery. The progression of constriction in an artery is caused by the boundaries irregularities. The flow of blood is discussed by many authors theoretically, experimentally as well as numerically for better understanding of flow properties in the presence of constriction. Forrester and Young [1] presented the analytical solution of Newtonian fluid for the axisymmetric, steady, incompressible flow and consider the mild constriction for the flow of blood both theoretically and experimentally in the converging and diverging tube. Morgan and Young [2] examined the approximate analytical solution of axisymmetric, steady flow which is applicable to both mild and severe constriction by using an integral method, this work may be considered as an extension of [1]. K. Haldar [3] investigated the analysis of blood flow treating it as viscous fluid flowing through an axisymmetric artery having constriction of cosine shape. Chow and Soda [4] presented the analytical solution for Newtonian fluid flow in an axisymmetric tube valid for the case where the spread of roughness is large compared with mean radius of the tube. Chow *et al.* [5] analyzed the steady laminar flow of Newtonian fluid for different physical quantities by considering the sinusoidal wall surface variation.

Previous work found in literature and cited above is limited to the flow pattern, pressure gradient, separation and reattachment points. In present investigation the blood is assumed to be Newtonian fluid of constant density flowing through a symmetric artery with constriction of cosine shape and constant volume flow rate. In the present paper the stream line, wall shear stress, separation, reattachment points and temperature distribution of blood flowing through an artery are analyzed by two methods namely ADM and RPM. The results where possible are compared with published data and found good agreement.

3. PROBLEM FORMULATION

It is assumed that the blood behaves like a homogeneous, incompressible, non-isothermal Newtonian fluid and the flow field is independent of time. At the inlet and outlet sections of the artery the flow is assumed to be Poiseuille or fully developed flow. The axial direction of flow is considered along $\tilde{z}-\text{axis}$ and $\tilde{r}-\text{axis}$ is normal to it. The profile of constriction in dimensional form is

$$R(\tilde{z})=\begin{cases} R_o-\dfrac{\lambda}{2}\left(1+\cos\left(\dfrac{4\pi\tilde{z}}{l_o}\right)\right) & -\dfrac{l_o}{4}<\tilde{z}<\dfrac{l_o}{4}, \\ R_o & \text{otherwise}, \end{cases} \tag{1}$$

where λ is the maximum height of constriction, $l_o/2$ the length of constricted region and R_o, $R(\tilde{z})$ are the unobstructed and variable radius of the artery in the constricted region. According to geometry of the problem, the boundary conditions on velocity components are obtained as

$$\tilde{u}=\tilde{w}=0 \quad at \quad \tilde{r}=R(\tilde{z}),$$
$$\frac{\partial\tilde{w}}{\partial\tilde{r}}=0 \quad at \quad \tilde{r}=0,$$
$$\tilde{Q}=\int_0^{R(\tilde{z})}\tilde{r}\,\tilde{w}d\,\tilde{r}=\frac{1}{2}u_oR_o^2, \tag{2}$$

where first two boundary conditions are no slip conditions, third is symmetry, fourth is constant volume flow rate and u_o is the characteristic velocity. Boundary conditions on temperature distribution are

$$\tilde{T} = T_1 \qquad at \qquad \tilde{r} = R(\tilde{z}),$$
$$\frac{\partial \tilde{T}}{\partial \tilde{r}} = 0 \qquad at \qquad \tilde{r} = 0, \tag{3}$$

where T_1 is the temperature on the boundary. This is axisymmetric steady problem of blood flow in cylindrical coordinates and the velocity vector $\tilde{V}$ becomes as

$$\tilde{\bar{V}} = \left(\tilde{u}(\tilde{r},\tilde{z}), 0,\ \tilde{w}(\tilde{r},\tilde{z}),\ 0\right) \tag{4}$$

Introducing the dimensionless variables as follows

$$r = \frac{\tilde{r}}{R_o},\ z = \frac{\tilde{z}}{l_o},\ u = \frac{\tilde{u}}{u_o},\ w = \frac{\tilde{w}}{u_o},\ p = \frac{R_o^2}{\mu u_o l_o}\tilde{p},\ \theta = \frac{\tilde{T} - T_o}{T_1 - T_o}, \tag{5}$$

where T_o is the temperature of blood at the center.

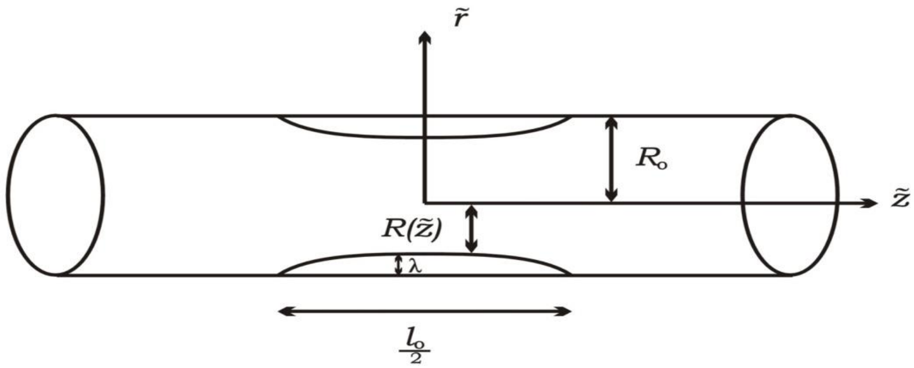

Figure 1: Geometry of the problem.

The dimensionless form of boundary profile (1) for the constricted region is

$$f(z) = 1 - \frac{\varepsilon}{2}\left(1 + \cos(4\pi z)\right) \qquad -\frac{1}{4} < z < \frac{1}{4},$$
$$= 1 \qquad \text{otherwise}, \tag{6}$$

where $f = \dfrac{R(\tilde{z})}{R_o}$ and $\varepsilon = \lambda / R_o$ are dimensionless measure of constriction in reference to artery.

The equations which govern the flow field are the conservation of mass and momentum along with the energy equation. These non-linear equations have unknown velocity components (u, v), pressure (p) and temperature (T).

Introducing the stream functions transformation to find the velocity components as

$$u = \frac{\delta}{r}\frac{\partial \psi}{\partial z},\ v = -\frac{1}{r}\frac{\partial \psi}{\partial r}, \tag{7}$$

which satisfying the continuity equation identically. After eliminating pressure gradient from momentum equations, the compatibility equation is obtained of the form

$$\mathrm{Re}\,\delta \frac{\partial\left(\psi, \frac{E^2\psi}{r^2}\right)}{\partial(z, r)} = \frac{1}{r}E^4\psi, \tag{8}$$

and energy equation in terms of stream function reduces as

$$\mathrm{Pe}\frac{\delta}{\mathrm{r}}\frac{\partial(\theta,\psi)}{\partial(r,z)} = \nabla^2\theta + \mathrm{Br}\left[2\delta^2\left\{\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial z}\right)\right)^2 + \left(\frac{1}{r^2}\frac{\partial \psi}{\partial z}\right)^2 + \left(\frac{1}{r}\frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial r}\right)\right)^2\right\} + \left(\frac{\delta^2}{r}\frac{\partial^2 \psi}{\partial z^2} - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right)\right)^2\right], \tag{9}$$

The boundary conditions in terms of stream function becomes (10)

$$-\frac{1}{r}\frac{\partial \psi}{\partial r} = 0, \qquad \psi = -\frac{1}{2},\ \theta = 1 \qquad at \qquad r = f,$$
$$-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) = 0, \quad \psi = 0, \quad \frac{\partial \theta}{\partial r} = 0 \qquad at \qquad r = 0.$$

where

$$E^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \delta^2\frac{\partial^2}{\partial z^2}, \qquad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \delta^2\frac{\partial^2}{\partial z^2} \tag{11}$$

and

$$\delta=\frac{R_o}{l_o},\quad \mathrm{Re}=\frac{u_oR_o}{\nu},\quad \mathrm{Br}=\frac{u_o^2\mu}{k(T_1-T_o)},\ \mathrm{Pe}=\frac{\rho u_oR_oc_p}{k}, \tag{12}$$

in which Re is the Reynolds number, Br the Brinkman number, Pe the Peclet number and δ the ratio of radius of the artery to the length of constriction.
Now our aim is to solve the compatibility equation (8) and energy equation (9) along with boundary conditions (10).

3. Solution of the Problem:

The resulting compatibility and energy equations are non-linear and finding of exact solution is very difficult. We apply ADM and RPM on these equations to find the series solution and considering δ as a small parameter for RPM. In the current investigations, we present the solution of non-linear equations by ADM and graphical discussion by both methods.

3.1 Solution of compatibility equation by ADM:

The compatibility equation (8) in terms of linear operator (L) and Adomian polynomial (A_n) is

$$L^2\psi=\delta A_n-\delta^4\frac{\partial^4\psi}{\partial z^4}-2\delta^2\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\partial^2\psi}{\partial z^2}, \tag{13}$$

where L and A_n are defined as

$$L=\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}, \tag{14}$$

$$A_n=r\,\mathrm{Re}\frac{\partial\left(\psi,\frac{E^2\psi}{r^2}\right)}{\partial(z,\,r)}, \tag{15}$$

Assuming that inverse operator L^{-1}. exist and defined as

$$L^{-1}(*)=\int\left(r\int\left(\frac{*}{r}\right)dr\right)dr.. \tag{16}$$

Operating L^{-2} to equation (13), we get

$$L^{-2}\left(L^2\psi\right)=\delta L^{-2}\left(A_n\right)-\delta^4L^{-2}\left(\frac{\partial^4\psi}{\partial z^4}\right)-2\delta^2L^{-2}\left\{\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\partial^2\psi}{\partial z^2}\right\}, \tag{17}$$

which gives

$$\psi=A\frac{r^4}{16}+B\left(\frac{r^2\ln r}{2}-\frac{r^2}{4}\right)+C\frac{r^2}{2}+D+\beta\left[\delta L^{-2}\left(A_n\right)-\delta^4L^{-2}\left(\frac{\partial^4\psi}{\partial z^4}\right)-2\delta^2L^{-2}\left\{\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\partial^2\psi}{\partial z^2}\right\}\right], \tag{18}$$

where β is dummy variable, which is used just to constitute the systems for different orders and A, B, C and D are functions of z to be determined. The systems for different orders of β are obtained by substituting

$$\psi=\sum_{n=0}^{\infty}\beta^n\psi_n(r,\,z),\quad A_n=\sum_{n=0}^{\infty}\beta^nA_n, \tag{19}$$

in equation (18), we obtain

$$\sum_{n=0}^{\infty}\beta^n\psi_n(r,z)=A\frac{r^4}{16}+B\left(\frac{r^2\ln r}{2}-\frac{r^2}{4}\right)+C\frac{r^2}{2}+D$$
$$+\beta\left[\delta L^{-2}\left(\sum_{n=0}^{\infty}\beta^nA_n\right)-\delta^4L^{-2}\left(\frac{\partial^4\left(\sum_{n=0}^{\infty}\beta^n\psi_n(r,z)\right)}{\partial z^4}\right)-2\delta^2L^{-2}\left\{\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\partial^2\left(\sum_{n=0}^{\infty}\beta^n\psi_n(r,z)\right)}{\partial z^2}\right\}\right], \tag{20}$$

and boundary conditions becomes

$$-\frac{1}{r}\frac{\partial\left(\sum_{n=0}^{\infty}\beta^n\psi_n(r,z)\right)}{\partial r}=0,\qquad \sum_{n=0}^{\infty}\beta^n\psi_n(r,z)=-\frac{1}{2}\qquad at\qquad r=f,$$
$$-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\left(\sum_{n=0}^{\infty}\beta^n\psi_n(r,z)\right)}{\partial r}\right)=0,\quad \sum_{n=0}^{\infty}\beta^n\psi_n(r,z)=0,\qquad at\qquad r=0. \tag{21}$$

Equating the power of β^0 on both sides of (20) and (21), we get

$$\psi_0 = A\frac{r^4}{16} + B\left(\frac{r^2 \ln r}{2} - \frac{r^2}{4}\right) + C\frac{r^2}{2} + D \qquad (22)$$

The boundary conditions for ψ_0 are

$$-\frac{1}{r}\frac{\partial \psi_o}{\partial r} = 0, \qquad \psi_o = -\frac{1}{2}, \qquad at \qquad r = f,$$

$$-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_o}{\partial r}\right) = 0, \quad \psi_o = 0, \qquad at \qquad r = 0. \qquad (23)$$

Applying these boundary conditions to (22), we obtain the solution for ψ_0 as

$$\psi_0 = \frac{\eta^2}{2}\left(\eta^2 - 2\right), \qquad where \qquad \eta = \frac{r}{f}, \qquad (24)$$

It is observed that the expression for ψ_0 is same by both the methods. The recursive relation for different orders of β becomes as

$$\psi_{n+1} = \delta L^{-2} A_n - \delta^4 L^{-2}\left(\frac{\partial^4 \psi_n(r,z)}{\partial z^4}\right) - 2\delta^2 L^{-2}\left\{\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\partial^2 \psi_n(r,z)}{\partial z^2}\right\}, \qquad (25)$$

Using the recursive relation (25), the relations for ψ_1 and ψ_2 along with boundary conditions comes out of the form

$$\psi_1 = \delta L^{-2} A_o - \delta^4 L^{-2}\left(\frac{\partial^4 \psi_o}{\partial z^4}\right) - 2\delta^2 L^{-2}\left\{\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\partial^2 \psi_o}{\partial z^2}\right\}, \qquad (26)$$

subject to boundary conditions

$$-\frac{1}{r}\frac{\partial \psi_1}{\partial r} = 0, \qquad \psi_1 = 0, \qquad at \qquad r = f,$$

$$-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_1}{\partial r}\right) = 0, \quad \psi_1 = 0, \qquad at \qquad r = 0. \qquad (27)$$

and

$$\psi_2 = \delta L^{-2} A_1 - \delta^4 L^{-2}\left(\frac{\partial^4 \psi_1}{\partial z^4}\right) - 2\delta^2 L^{-2}\left\{\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\partial^2 \psi_1}{\partial z^2}\right\}, \qquad (28)$$

along with boundary conditions

$$-\frac{1}{r}\frac{\partial \psi_2}{\partial r} = 0, \qquad \psi_2 = 0, \qquad at \qquad r = f,$$

$$-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_2}{\partial r}\right) = 0, \quad \psi_2 = 0, \qquad at \qquad r = 0. \qquad (29)$$

where

$$A_o = r \operatorname{Re} \frac{\partial\left(\psi_o, \frac{E^2 \psi_o}{r^2}\right)}{\partial(z, r)}, \qquad (30)$$

$$A_1 = r \operatorname{Re}\left\{\frac{\partial\left(\psi_o, \frac{E^2 \psi_1}{r^2}\right)}{\partial(z, r)} + \frac{\partial\left(\psi_1, \frac{E^2 \psi_o}{r^2}\right)}{\partial(z, r)}\right\}, \qquad (31)$$

The solution for ψ_1 is obtained from (26) by substituting ψ_o and operating L^{-2} subject to boundary conditions (27) as

$$\psi_1 = \frac{\delta \eta^2 \left(\eta^2 - 1\right)^2}{2880 f}\left[\begin{array}{l} 20 \operatorname{Re} \delta^2 f'^3 \left(3\eta^4 - 10\eta^2 + 13\right) - 150\delta^3 f f'^4\, f''\left(7\eta^2 + 2\right) + 60\delta f f'^2 \left(3\delta^2 f f''\left(5\eta^2 - 2\right) - 40\right) \\ + f'\left\{80 \operatorname{Re}\left(\eta^2 - 4\right) - \operatorname{Re} \delta^2 f\, f''\left(39\eta^4 - 142\eta^2 + 217\right) - 20\delta^3 f^3 f'''\left(5\eta^2 - 8\right)\right\} \\ + \delta f^2\left\{480 f'' - 15\delta^2 f\, f''^2 \left(5\eta^2 - 8\right) + \delta\left(\operatorname{Re} f'''\left(3\eta^4 - 14\eta^2 + 29\right) + 5\delta f^2 f^{(4)}\left(\eta^2 - 4\right)\right)\right\} \end{array}\right]. \qquad (32)$$

Similarly the solution for ψ_2 is obtained by substituting ψ_1 and ψ_o in equation (28) along with respective boundary conditions from (29) up to second order in δ is

$$\psi_2 = \frac{\mathrm{Re}^2\,\delta^2\eta^2(\eta^2-1)^2}{21600\,f^2}\left[f'^2\left(38\eta^6-314\eta^4+759\eta^2-818\right)-3f\,f''\left(2\eta^6-16\eta^4+41\eta^2-52\right)\right] \quad (33)$$
$$+\ higher\quad power\quad of\quad \delta.$$

Now one can obtain the solution for ψ upto second order by using $\psi = \psi_o + \psi_1 + \psi_2$ and velocity components u and w from equation (7).

3.2 Solution of Energy Equation by ADM

The energy equation (9) in terms of linear operator L_1 is

$$L_1\theta = \mathrm{Pe}\frac{\delta}{\mathrm{r}}\frac{\partial(\theta,\psi)}{\partial(r,z)} - \mathrm{Br}\left[\begin{array}{l} 2\delta^2\left\{\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)\right)^2+\left(\frac{1}{r^2}\frac{\partial\psi}{\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial r}\right)\right)^2\right\} \\ +\left(\frac{\delta^2}{r}\frac{\partial^2\psi}{\partial z^2}-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right)^2 \end{array}\right] - \delta^2\frac{\partial^2\theta}{\partial z^2}, \quad (34)$$

subject to boundary conditions

$$\theta = 1 \quad at \quad r = f \quad and \quad \frac{\partial\theta}{\partial r} = 0 \quad at \quad r = 0. \quad (35)$$

where L_1 is defined as

$$L_2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}, \quad (36)$$

and assume that inverse operator exist and defined as

$$L_1^{-1}(*) = \int\left(\frac{1}{r}\int r\,(*)dr\right)dr. \quad (37)$$

Apply L_1^{-1} to equation (34), we obtain

$$\theta = C_1\ln r + C_2 + \beta L_1^{-1}\left[\mathrm{Pe}\frac{\delta}{\mathrm{r}}\frac{\partial(\theta,\psi)}{\partial(r,z)} - \mathrm{Br}\left[\begin{array}{l} 2\delta^2\left\{\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)\right)^2+\left(\frac{1}{r^2}\frac{\partial\psi}{\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial r}\right)\right)^2\right\} \\ +\left(\frac{\delta^2}{r}\frac{\partial^2\psi}{\partial z^2}-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right)^2 \end{array}\right] - \delta^2\frac{\partial^2\theta}{\partial z^2}\right], \quad (38)$$

where C_1 and C_2 are functions of z to be determined. Now for the different orders of θ substituting

$$\theta = \sum_{n=0}^{\infty}\beta^n\theta_n(r,z), \quad \psi = \sum_{n=0}^{\infty}\beta^n\psi_n(r,z),, \quad (39)$$

in equation (38) and comparing both sides, we get

$$\theta_o = C_1\ln r + C_2,, \quad (40)$$

and boundary conditions are

$$\theta_o = 1 \quad at \quad r = f \quad and \quad \frac{\partial\theta_o}{\partial r} = 0 \quad at \quad r = 0. \quad (41)$$

The solution for (40) subject to boundary conditions (41) is

$$\theta_o = 1. \quad (42)$$

It is observed that the solution obtained in equation (42) is independent of Br number, while the zeroth order perturbation solution involves Br number. Now for the first and second order temperature equating the coefficients of β and β^2 from (38) with respective boundary conditions from (35) as follows

$$\theta_1 = L_1^{-1}\left[\mathrm{Pe}\frac{\delta}{\mathrm{r}}\frac{\partial(\theta_o,\psi_o)}{\partial(r,z)} - \mathrm{Br}\left[\begin{array}{l} 2\delta^2\left\{\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_o}{\partial z}\right)\right)^2 + \left(\frac{1}{r^2}\frac{\partial\psi_o}{\partial z}\right)^2 + \left(\frac{1}{r}\frac{\partial}{\partial z}\left(\frac{\partial\psi_o}{\partial r}\right)\right)^2\right\} \\ + \left(\frac{\delta^2}{r}\frac{\partial^2\psi_o}{\partial z^2} - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_o}{\partial r}\right)\right)^2 \end{array}\right] - \delta^2\frac{\partial^2\theta_o}{\partial z^2}\right], \tag{43}$$

with boundary conditions

$$\theta_1 = 1 \quad at \quad r = f \quad and \quad \frac{\partial\theta_1}{\partial r} = 0 \quad at \quad r = 0. \tag{44}$$

$$\theta_2 = L_1^{-1}\left[\begin{array}{l} \mathrm{Pe}\frac{\delta}{\mathrm{r}}\left(\frac{\partial(\theta_o,\psi_1)}{\partial(r,z)} + \frac{\partial(\theta_1,\psi_o)}{\partial(r,z)}\right) - \delta^2\frac{\partial^2\theta_1}{\partial z^2} \\ -2\mathrm{Br}\left[\begin{array}{l} 2\delta^2\left\{\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_o}{\partial z}\right)\right)\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_1}{\partial z}\right)\right) + \left(\frac{1}{r^2}\frac{\partial\psi_o}{\partial z}\right)\left(\frac{1}{r^2}\frac{\partial\psi_1}{\partial z}\right) + \left(\frac{1}{r}\frac{\partial}{\partial z}\left(\frac{\partial\psi_o}{\partial r}\right)\right)\left(\frac{1}{r}\frac{\partial}{\partial z}\left(\frac{\partial\psi_1}{\partial r}\right)\right)\right\} \\ + \frac{\delta^4}{r^2}\frac{\partial^2\psi_o}{\partial z^2}\frac{\partial^2\psi_1}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_o}{\partial r}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_1}{\partial r}\right) - \frac{\delta^2}{r}\frac{\partial^2\psi_o}{\partial z^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_1}{\partial r}\right) - \frac{\delta^2}{r}\frac{\partial^2\psi_1}{\partial z^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi_o}{\partial r}\right) \end{array}\right] \end{array}\right], \tag{45}$$

subject to boundary conditions

$$\theta_2 = 1 \quad at \quad r = f \quad and \quad \frac{\partial\theta_2}{\partial r} = 0 \quad at \quad r = 0. \tag{46}$$

The solution of (43) along with boundary conditions (44) becomes

$$\theta_1 = -\frac{Br\left(\eta^2-1\right)}{144 f^4}\left\{\begin{array}{l} 144\left(\eta^2+1\right) + 3\delta^4 f'^4\left(75\eta^6 - 85\eta^4 + 23\eta^2 + 23\right) + 16\delta^2 ff''\left(4\eta^4 - 5\eta^2 - 5\right) \\ + \delta^4 f^2 f''^2\left(9\eta^6 - 23\eta^4 + 13\eta^2 + 13\right) + 2\delta^2 f'^2\left(\begin{array}{l} 8\left(32\eta^4 - 49\eta^2 + 59\right) \\ -\delta^2 ff''\left(45\eta^6 - 83\eta^4 + 25\eta^2 + 25\right) \end{array}\right) \end{array}\right\}, \tag{47}$$

which depends upon ratio of heat production by viscous dissipation to heat transport by conduction. The solution of (45) along with boundary conditions (46) up to second order is

$$\begin{aligned} \theta_2 &= \delta\left[-\frac{Br\left(\eta^2-1\right)f'}{72 f^5}\left\{4\,\mathrm{Re}\left(3\eta^6 - 13\eta^4 + 5\eta^2 + 5\right) + Pe\left(9\eta^6 - 7\eta^4 - 43\eta^2 + 101\right)\right\}\right] \\ &+ \delta^2\left[\frac{Br\left(\eta^2-1\right)}{9 f^4}\left\{f'^2\left(54\eta^4 - 2\eta^2 - 47\right) - f\, f''\left(10\eta^4 - 2\eta^2 - 11\right)\right\}\right] + higher\ power\ of\ \delta. \end{aligned} \tag{48}$$

which depends upon the heat production by viscous dissipation to heat transport and heat transport by convection to conduction.

4 Wall Shear Stress:

The dimensionless form of wall shear stress for viscous fluid is given by

$$\tau_\omega = \delta\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \tag{49}$$

Substituting the velocity components correct up to the second order in δ, the wall shear stress is obtained of the form

$$\tau_\omega = \frac{4f'}{f^3} - \frac{2\,\mathrm{Re}\,\delta\, f'^2}{3f^4} + \frac{\delta^2 f'}{f^5}\left(4f^2 f' - \frac{67\,\mathrm{Re}^2\ f'^2}{540} - \frac{20\, f^2\ f'^2}{3} + \frac{\mathrm{Re}^2\ f\ f''}{36} + \frac{4f^3\ f''}{3}\right). \tag{50}$$

The points of separation and reattachment at the wall are those points where the reverse flow occurs and obtained by setting the wall shear stress equal to zero i.e, $\tau_{\omega} = 0$. The resulting equation is quadratic in Re, the solution in terms of Reynolds number is

$$\mathrm{Re} = \frac{12}{\delta\left(15 f f'' - 67 f'^2\right)}\left\{15 f\, f' \pm \sqrt{5\left(45 f^2 f'^2 - \left(15 f f'' - 67 f'^2\right)\left(3 f^2 + \delta^2 f^2\left(3 f' - 5 f'^2 + f\, f''\right)\right)\right)}\right\}, \quad (51).$$

now from the above relation, our task is to find the critical Reynolds number graphically where the separation and reattachment points occurs. The expression for the pressure gradient up to third order in δ is obtained by using the component form of momentum equation as follows

5. GRAPHICAL DISCUSSION:

In this section effect of various parameters that control the fluid flow are discussed graphically. In figure 2, the stream lines for the flow pattern are shown by taking z-component parallel to the axis of the artery and r-component perpendicular to the axis of artery. It is observed that both the method gives similar presentation of stream lines. The zeroth order stream lines are presented in figure 2(a) by ADM and RPM. It is observed that the stream lines are relatively straight lines at the center for ε = 0.20, Re = 12, δ = 0.1 by both the methods and first order solution is depicted in figure 2(b) by ADM and RPM induces the clockwise and counterclockwise rotational motion in the converging and diverging regions of the artery. It is found from first order solution that the separation point lays in the converging region of the artery and reattachment point lies in the diverging region. Figure 2(c) presents graphically the flow pattern for the second order stream lines by ADM and RPM, which reinforce the first order solution and shows the rotational motion indicating the separation and reattachment points. Figures 2(d) shows the flow pattern of streamlines up to the second order in δ by ADM and RPM, for the fixed values of the parameters in the converging and diverging regions of the artery. It is observed that the stream lines are relatively straight at the center.

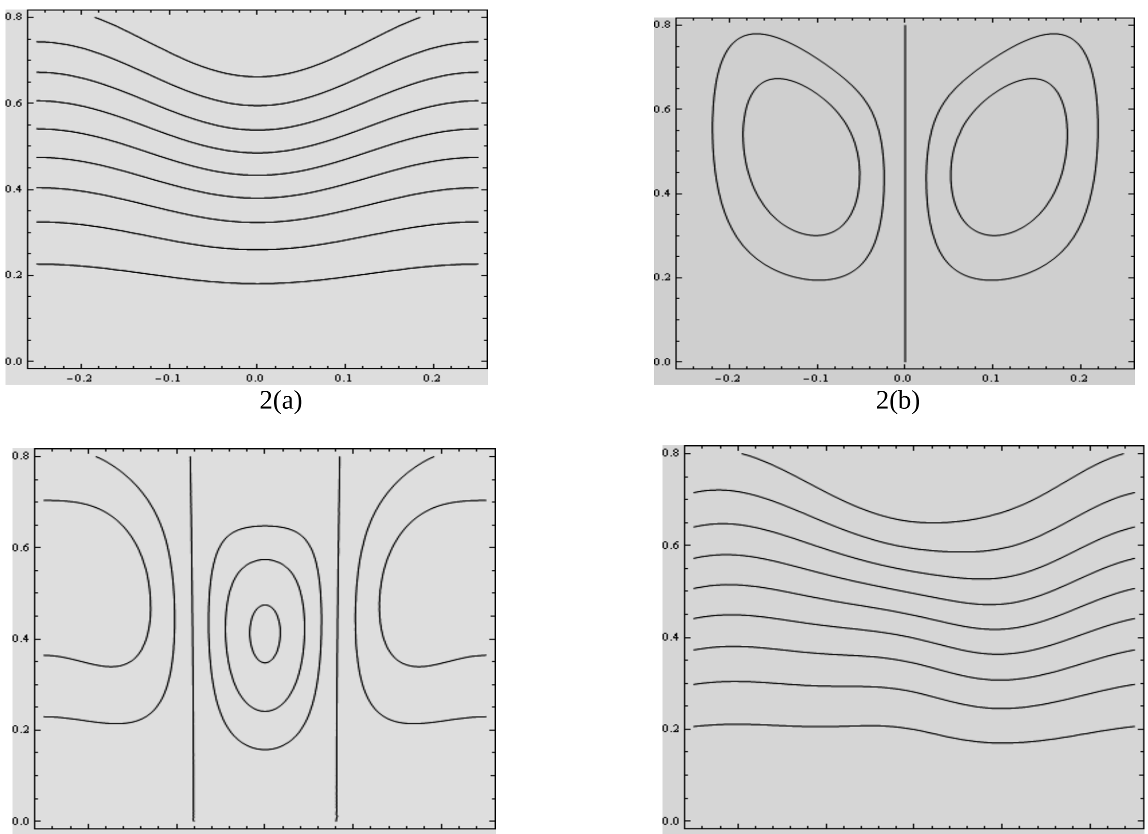
2(a) 2(b) 2(c) 2(d)

Fig. 2 The stream lines for the flow pattern

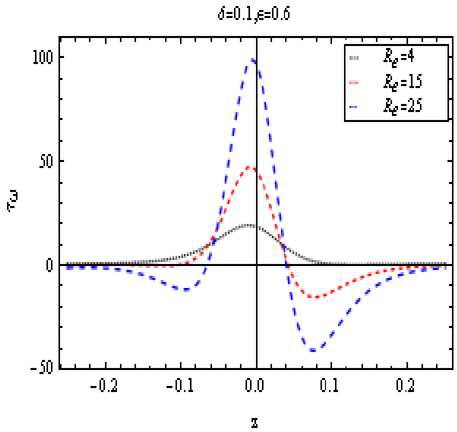

Fig.3 Effect of Re on wall shear stress by ADM.

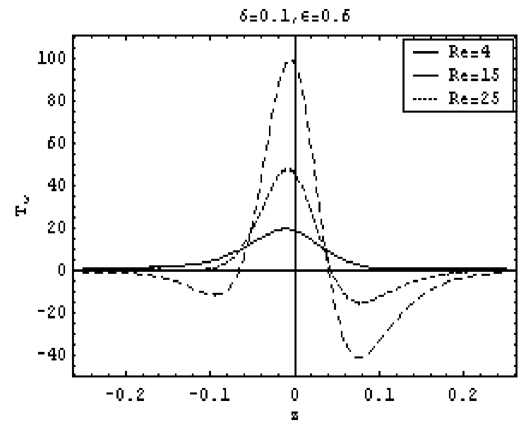

Fig.4 Effect of Re on wall shear stress by RPM.

Figures 3 and 4 depicts the effect of Re on wall shear stress by using fixed values as ε = 0.6 and δ = 0.1 for ADM and RPM respectively. Wall shear stress increases as the Re increases near the throat of the artery and becomes negative in the converging and diverging sections of the artery. The negative shearing indicates the occurrence of separation point in the converging section and reattachment point in the diverging section of the artery. Figures 5 and 6 present the effect of ε on wall shear stress by ADM and RPM respectively. It is observed that the flow indicates the Poiseuille flow in the absence of constriction. It is also observed that with the increase in height of constriction wall shear stress increases and become negative in the converging and diverging sections of the artery which are due to adverse flow and indicates the points of separation and reattachment.

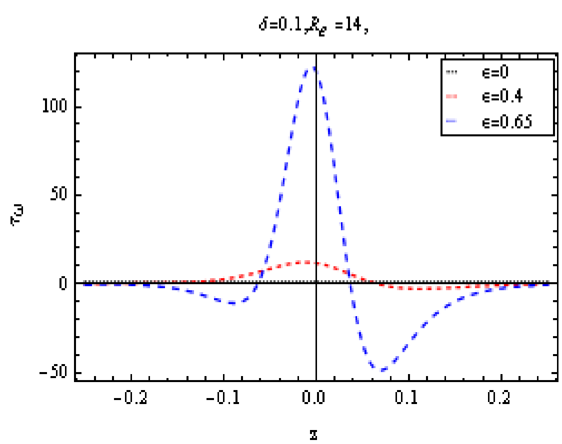

Fig.5 Effect of ε on wall shear stress by ADM.

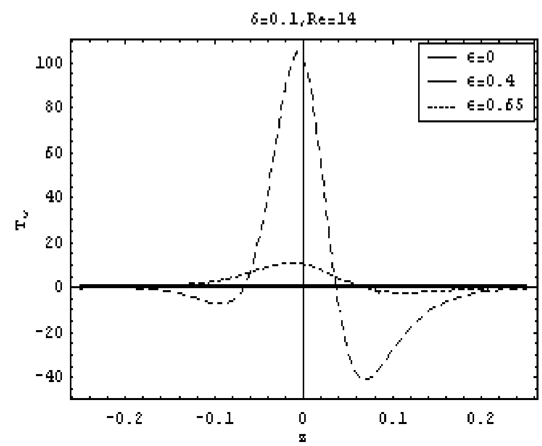

Fig.6 Effect of ε on wall shear stress by RPM.

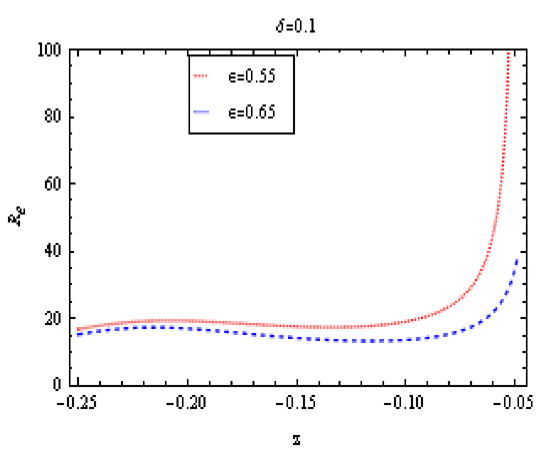

Fig.7 Separation points in converging region by ADM.

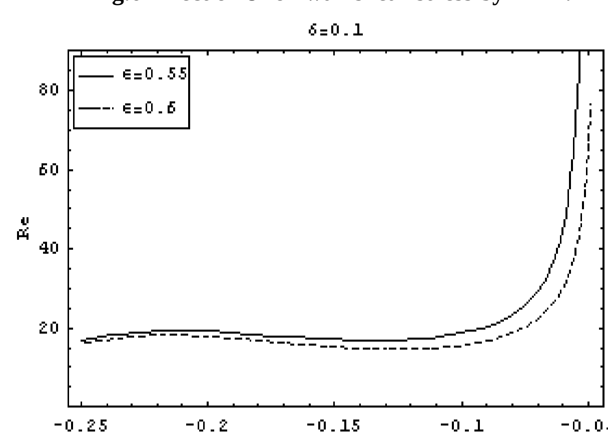

Fig.8 Separation points in converging region by RPM.

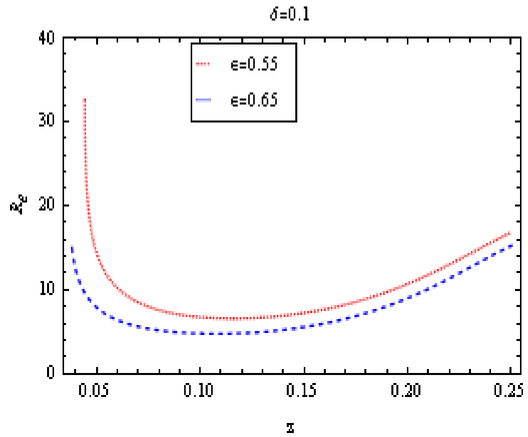

Fig.9 Reattachment points in diverging region by ADM.

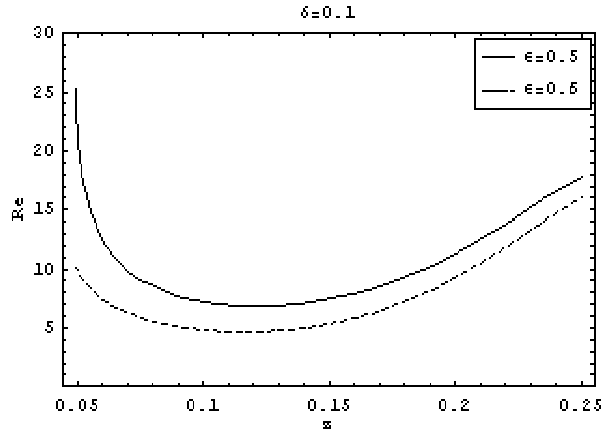

Fig.10 Reatt. points in diverging region by RPM.

Figures 7 and 8 present the phenomena of separation point in the converging region by ADM and RPM respectively. It is found that separation point occurs as there is no wall shear stress i.e., zero wall shear stress. In this analysis our aim is to find the critical Re at which the separation occurs. It is observed that with the increase in ε critical Re decreases in the converging section of the artery. Figures 9 and 10 depict the reattachment point in the diverging region by ADM and RPM. It is found that by the increase in ε critical Reynolds number Re decreases with the fixed value of δ = 0.1.

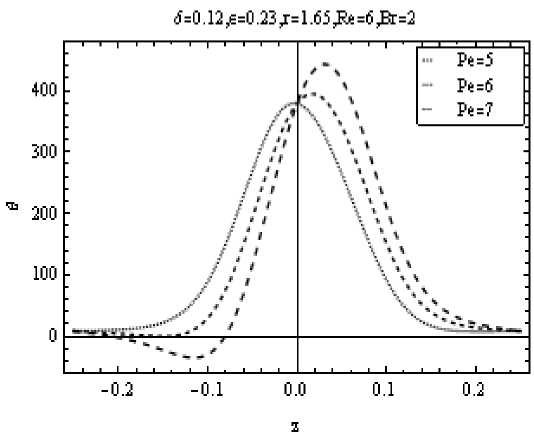

Fig.11 Effect of Pe on temperature distribution by ADM.

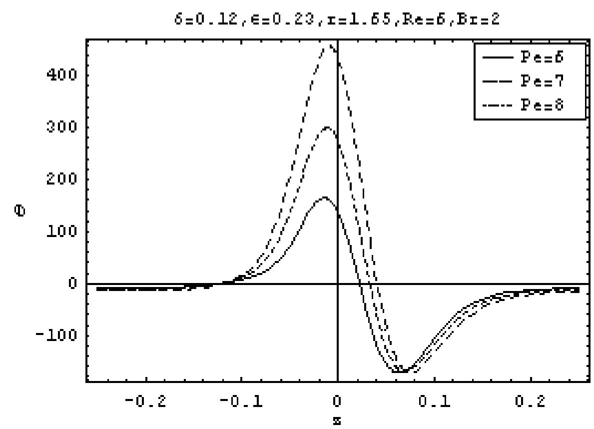

Fig.12 Effect of Pe on temp. distribution by RPM.

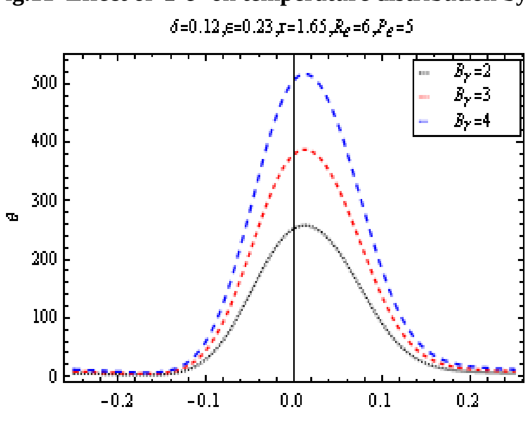

Fig.13 Effect of Br on temperature distribution by ADM.

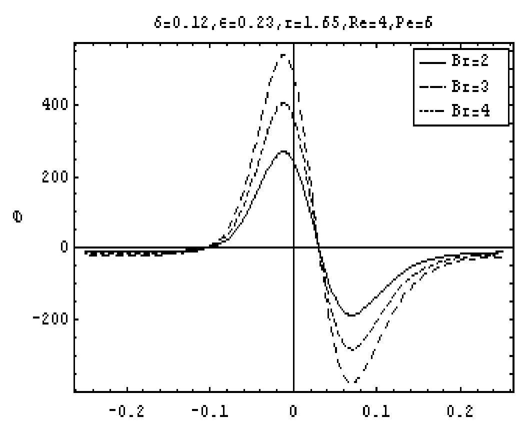

Fig.14 Effect of Br on temp. distribution by RPM.

Figures 11 and 12 present the effect of Pe on temperature distribution by ADM and RPM respectively. It is observed that by the increase in Pe along with the fixed values of the other parameters, the temperature increases over the constriction and becomes negative in the converging and diverging regions due to reverse flow. Similarly figure 13 and 14 shows the behavior of Br on temperature distribution by ADM and RPM respectively, for the fixed values of various parameters. It is found that as increase in Br increases the temperature on the constricted region and becomes negative in the converging and diverging regions of the artery due to adverse flow.

6. CONCLUSION

The analysis of steady state blood flow with heat transfer in an axisymmetric artery having constriction of cosine shape is presented. It is assumed that the blood behaves like the homogeneous and incompressible Newtonian fluid in an artery. The governing equations are transformed into stream function and solved analytically with the help of Adomian decomposition method and Regular perturbation technique. The solutions thus obtained are compared for velocity components, wall shear stress, separation and reattachment points and temperature distribution by two methods. Further discussion has been carried out on the impact of above mentioned quantities. It is observed that the solutions obtained in the analysis are comparable with the results existing in the literature. The general pattern of streamlines is same as [4] – [5], wall shear stress is similar as [2] – [3] and separation and reattachment points are identical with [3]. Conclusions made from the above investigations are as follows

(i) Increase in Re increases the velocity of blood, wall shear stress and temperature.

(ii) Increase in ε increases the velocity of blood, wall shear stress and temperature.

(iii) Increase in ε decreases the critical Re.

(iv) Increase in Br and Pe increases the temperature.

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